Some Information on Helmholtz Coils

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1 Magnetic Field of a Loop of Wire

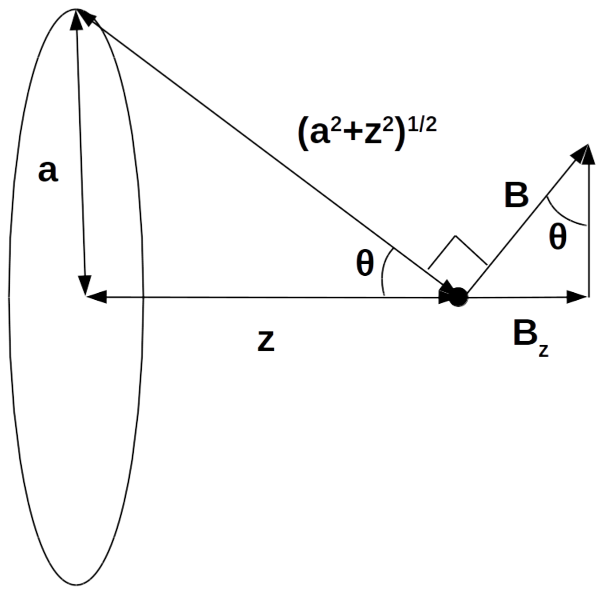
A Helmholtz coil is composed of two loops of wire. Therefore, let's start by calculating the magnetic field of one loop of wire.

Let's align this coil in the xy plane and calculate the magnetic field along the z-axis.

The x and y components must vanish by symmetry,[1] so we only have to calculate the z component.

Let's let our coil have radius a, and look at it some distance z away.

[1] In a bit more detail: suppose you rotate the ring in the xy plane, about the z-axis. On the one hand, you rotated the thing making the magnetic field, so you have to rotate the magnetic field vector, too. On the other hand, you didn't change the actual configuration, so the magnetic field vector has to stay the same. Therefore, the magnetic field has to be the same when it's rotated about the z-axis, which means it cannot have an x or y component



We will use the formula:

$$B = \frac{\mu_0 IL \times \hat{r}}{4\pi r^2} \tag{1}$$

Here, r the distance between the wire and the point of interest, and $\hat{r}$ the unit vector in that direction. This determines the magnetic field of a short length of wire L (short enough that r and $\hat{r}$ don't change too much).[2]

Then we will note is that B_z is the same for any part of the wire, so in fact r doesn't change at all. Neither does the z-component of $\hat{r}$. This means that the above formula works exactly for the z-component.

Some knowledge of the cross product allows us to understand that we can make the replacement $IL \times \hat{r} \rightarrow IL\sin(\theta)$: because we only care about the part of B along the z-axis, we only care about the part of $\hat{r}$ perpendicular to the z-axis, which is precisely $\sin(\theta)$ (since the magnitude of $\hat{r}$ is 1, by definition).

[2]This is really a 132 formula in general, as it's used much more (combined with a little calculus) to derive the magnetic field for more complicated arrangements (like a finite wire, and thereby a *square* loop of wire, as opposed to a circular one). Here, you get just a little glimpse of its power!

Knowing that $\sin(\theta) = \frac{a}{\sqrt{z^2+a^2}}$ (from the above diagram) and $L = 2\pi a$ (as the circumference), we can then calculate the resultant magnetic field:

$$B_z = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}} \tag{2}$$

This is the formula for the magnetic field of a single loop along the z-axis. If we add more loops, we get an extra factor of N:

$$B_z = \frac{\mu_0 N I a^2}{2(a^2+z^2)^{3/2}} \tag{3}$$

If you plug in $z = 0$, you get the formula you are most likely familiar with (from your textbook, and your equation sheet) for the magnetic field at the center of a coil:

$$B_z = \frac{\mu_0 N I}{2a}$$

2 Magnetic Field of a Helmholtz Coil

Having done that, we're now past the hard part. A Helmholtz coil, after all, is just two loops put together, so we can just add the magnetic fields of the two currents together (just like you do for electric fields of two charges - this is generally called the *law of superposition*).

Let's start by having our loops separated by an arbitrary distance d. That means that the second coil is going to have the difference in z position from it actually be $d - z$. This makes our full equation:

$$\begin{aligned} B_z = & \frac{\mu_0 N I a^2}{2(a^2+z^2)^{3/2}} + \frac{\mu_0 N I a^2}{2(a^2+(d-z)^2)^{3/2}} \\ = & \frac{\mu_0 N I a^2}{2}\left((a^2+z^2)^{-3/2} + (a^2+(d-z)^2)^{-3/2}\right) \end{aligned} \tag{4}$$

In the second version, we just factored out all the stuff that was in common.

Plugging in $z = a/2$ and $d = a$ (and doing some algebra) gives us the equation for the magnetic field at the center of a Helmholtz coil:

$$B_z = \frac{8}{5\sqrt{5}} \frac{\mu_0 N I}{a} \tag{5}$$

3 Usefulness of Helmholtz Coils

Now, let's look at why Helmholtz coils are useful: they make "mostly flat" magnetic fields. (Warning: this section will involve some calculus, in particular Taylor expansions, although we won't be working through the computational details.)

Let's suppose we Taylor-expand B_z about the center of the coil, $z = d/2$ (for a general d, again - we're not considering). Let B_0 be the value at the center (we don't care what it is for the moment).

If you crunch out enough calculus and algebra (or just tell Wolfram—Alpha to Taylor-expand it for you), you end up with the expression:

$$B_z \simeq B_0 - \frac{96\mu_0 N I a^2}{(4a^2 + d^2)^{7/2}}(a^2 - d^2)(x - d/2)^2 + O(x^4) \tag{6}$$

Thus, the magnetic field is B_0 plus some quadratic-order correction (the odd terms vanish by symmetry).

However, look at the second part of that quadratic term. If we set $a = d$ - that is to say, we make our arbitrary coil a Helmholtz coil - then that coefficient vanishes as well, and our approximation is valid to *fourth* order. That is to say, the magnetic field is more flat at the center with this arrangement than otherwise.

There are improvements that try to minimize the deviation over the whole range and other more practical goals - slight tweaks that bring the error down in practice. But those are, in a sense, details; the essential part of understanding why Helmholtz coils are naturally good is that Taylor approximation.

In fact, there's a further improvement - Maxwell coils - which vanish to sixth order, by adding a third coil. Same idea - more complicated computationally, but useful for more precise experiments.